VARIATIONS OF MASS FORMULAS FOR DEFINITE DIVISION ALGEBRAS

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Abstract. In this article we study all possible masses arising from a definite central division algebra over a global field. We determine the relations among these masses and evaluate them explicitly. These formulas are useful in estimating the size of class numbers; they can be also used to determine some special definite orders of class number one.

1. Introduction

Let K be a global field and S be a non-empty finite set of places of K that contains all Archimedean places if K is a number field. Let A be the ring of S-integers. If K is a number field and S is the set of all Archimedean places, then A is simply the ring of integers. When K is a function field, A is the ring of functions in K regular away from S.

Let D be a fixed definite central division algebra D of degree n over K with respect to S. That is, every completion $D_v := D \otimes_K K_v$ at $v \in S$ is a central *division* algebra over K_v, or D does not satisfy the Eichler condition relative to S (by definition). Let R be any A-order in D. Let G be the multiplicative group of D, viewed as a group scheme over A with the integral structure endowed by R. Let G_1 denote the reduced norm one subgroup scheme of G, and let $G^{\rm ad} := G/Z$ denote the adjoint group scheme of G. Put $U := \widehat{R}^\times \subset G(\mathbb{A}^S)$, where $\widehat{R} = \prod_{v \notin S} R_v$ is the profinite completion of R and $\mathbb{A}^S$ is the prime-to-S adele ring of K. Put $U_1 := U \cap G_1(\mathbb{A}^S)$ and $U^{\rm ad} := U/\widehat{A}^\times \subset G^{\rm ad}(\mathbb{A}^S)$, where $\widehat{A} = \prod_{v \notin S} A_v$ is the profinite completion of A. Both groups $G_1(K)$ and $G^{\rm ad}(K)$ have S-arithmetic subgroups of finite order; see Section 2.1 for details. Therefore one can associate the masses $\mathrm{Mass}(G_1, U_1)$ and $\mathrm{Mass}(G^{\rm ad}, U^{\rm ad})$ group-theoretically; see Section 2.2 for the definition.

On the other hand, there is another more classical definition of the mass associated to the pair (D, R) using algebras, which may date from Deuring and Eichler; see [10], cf. [26]. Let $\{I_1, \dots, I_h\}$ be a complete set of representatives of the right locally principal ideal classes of R. Define the *mass of* (D, R) by

$$\mathrm{Mass}(D, R) := \sum_{i=1}^{h} [R_i^\times : A^\times]^{-1}, \tag{1.1}$$

where R_i is the left order of I_i. See Section 2.3 for detailed discussions.

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The main contents of this article are to compare these masses and evaluate them explicitly.

Our first main result is the following; see Theorem 3.2 and Corollary 3.9.

Theorem 1.1. *Let K, S, A, D, R be as above. We have*

$$\mathrm{Mass}(D,R) = h_A \cdot \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}), \tag{1.2}$$

where h_A is the class number of A. Moreover, we have

$$\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) = c(S,U) \cdot \mathrm{Mass}(G_1, U_1), \tag{1.3}$$

where

$$c(S,U) := \begin{cases} n^{-(|S|-1)}[\widehat{A}^\times : \mathrm{Nr}(U)] & \text{if } K \text{ is a function field;} \\ 2^{-(|S|-|\infty|-1)}[\widehat{A}^\times : \mathrm{Nr}(U)] & \text{if } K \text{ is a number field.} \end{cases} \tag{1.4}$$

Here $\mathrm{Nr} : G(\mathbb{A}^S) \to \mathbb{A}^{S,\times}$ denotes the reduced norm map.

The second main result evaluates the mass $\mathrm{Mass}(D,R)$ explicitly; see Theorem 4.2. By Theorem 1.1 we also obtain formulas for $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ and $\mathrm{Mass}(G_1, U_1)$.

Theorem 1.2. *Notations are as in Theorem 1.1. We have*

$$\mathrm{Mass}(D,R) = \frac{h_A}{n^{|S|-1}} \cdot \prod_{i=1}^{n-1} |\zeta_K(-i)| \cdot \prod_v \lambda_v(R_v), \tag{1.5}$$

where $\zeta_K(s)$ is the Dedekind zeta function of K, v runs through all non-Archimedean places of K and

$$\lambda_v(R_v) := \frac{|\mathfrak{d}(R_v)|}{|\kappa(R_v)^\times|/|\kappa(R_v)|} \cdot \prod_{1\le i\le n} (1-q_v^{-i}), \tag{1.6}$$

where $\mathfrak{d}(R_v)$ is the reduced discriminant of R_v, $|\mathfrak{m}_v|$ is the cardinality of $O_v/\mathfrak{m}_v$ for a non-zero integral ideal $\mathfrak{m}_v \subset O_v$ (the valuation ring in K_v), q_v is the cardinality of the residue field $\kappa(v)$ of K at v and $\kappa(R_v) := R_v/\mathrm{rad}(R_v)$ is the quotient of R_v by its Jacobson radical $\mathrm{rad}(R_v)$. Here we set R_v, for any non-Archimedean place $v \in S$, to be the unique maximal order in D_v.

The formula (1.5) of Theorem 1.2 agrees with Gross's formula [18, Proposition 10.7] in the number field case. Gross's formula relies on a formula for Artin conductors on tori, which was not yet known in the positive characteristic case. The conjectural formula was established later by C.-L. Chai and J.-K. Yu [5].

When K is the rational function field and S is the "place at infinity", that is, $A = \mathbb{F}_q[t]$ is the polynomial ring, the formula (1.5) for hereditary orders R is used in Denert and Van Geel [6] to prove the cancellation property for orders in definite central division algebras over $K = \mathbb{F}_q(t)$.

When $n = 2$, Theorem 1.2 gives the following formula (Corollary 4.3).

Corollary 1.3. *Let K, S, A, D, R be as above and assume that $n = 2$. Then we have*

$$\mathrm{Mass}(D,R) = \frac{h_A|\zeta_K(-1)|}{2^{|S|-1}} \prod_{v\in S_R} |\mathfrak{d}(R_v)| \frac{(1-q_v^{-2})}{(1-e(R_v)q_v^{-1})}. \tag{1.7}$$

where S_R consists of all non-Archimedean places v of K such that either v is ramified in D or R_v is not maximal, and $e(R_v)$ is the Eichler symbol (see § 4).

When K is a totally real number field and $S=\infty$, the set of Archimedean places of K, Corollary 1.3 is obtained first by Körner (see [20, Theorem 1], also see [19] for the computation). Körner's formula is used by Brzezinski [3] to solve the class number one problem for orders in all definite quaternion algebras over $\mathbb{Q}$.

After the explicit formulas have been established, one can easily check them using Gross or Prasad's general formulas (see [21, 18]) and computations in this paper. The formulas presented here originally are not derived from their formulas, but by rather elementary and explicit ways based on methods of Eichler [10] on quaternion algebras and results of Weil [28]; see [30]. For some people who like to apply mass formulas for some arithmetic and geometric applications (e.g. [3, 6, 13, 15, 29, 31, 32, 33, 34, 36]), it could be more convenient to have both explicit formulas and explicit computations for proofs at hand.

In the last part of this paper we define a mass for types of A-orders and evaluate it explicitly. More precisely, let $T(R)$ denote the set of $D^\times$-conjugacy classes of the genus $\mathcal{R}$ of R and let $\{R_1,\dots,R_t\}$ be a complete set of representatives of A-orders, where t is the type number of R. It is not hard to see that each index $[N(R_i):K^\times]$ is finite, where $N(R_i)$ is the normalizer of R_i in $D^\times$. We define

$$\mathrm{Mass}(T(R)) := \sum_{i=1}^{t} [N(R_i):K^\times]^{-1}. \tag{1.8}$$

The following theorem gives an explicit formula for $\mathrm{Mass}(T(R))$.

Theorem 1.4. *Notations as above. We have*

$$\mathrm{Mass}(T(R)) = \frac{1}{n^{|S|-1}}\cdot\prod_{i=1}^{n-1}|\zeta_K(-i)|\cdot\prod_v \lambda_v(R_v)\cdot[\mathcal{N}(\widehat{R}):\mathbb{A}^{S,\times}\widehat{R}^\times], \tag{1.9}$$

where the term $\lambda_v(R_v)$ is as in Theorem 1.2 and $\mathcal{N}(\widehat{R})$ is the normalizer of $\widehat{R}$ in $\widehat{D}^\times = G(\mathbb{A}^S)$.

As the mass formula gives an approximation for the size of class numbers, the formula (1.9) plays the similar role for the type numbers. This was not studied in the literature.

This paper is organized as follows. Section 2 discusses all possible definitions of masses arising from a definite central division algebra over a global field. Section 3 compares these masses and gives the explicit formula (Theorem 1.2) when R is a maximal order. In Section 4 we compute the local indices and obtain the mass formulas for an arbitrary order R. Last section discusses a mass for types of orders and gives an explicit mass formula.

2. Definitions of masses

2.1. **Setting.** Let K be a global field. Let S be a non-empty finite set of places of K that contains all Archimedean places if K is a number field or contains a fixed place ∞ if K is a global function field. We also write ∞ for the set of Archimedean places when K is a number field. Let A be the ring of S-integers. If K is a number field and $S=\infty$, then A is the ring of integers in K which is usually denoted by O_K. When K is a global function field, A is the ring of functions in K regular away from S. Let V^K denote the set of all places of K and denote by S^c the complement of S in V^K. There is a natural one-to-one bijection between the set S^c and the set

$\mathrm{Max}(A)$ of non-zero prime ideals of A. For any place v of K, let K_v denote the completion of K at v and O_v the valuation ring if v is non-Archimedean. If $v \notin S$, one also writes A_v for O_v, the completion of A at v. Let $\mathbb{A}$ be the adele ring of K, $\mathbb{A}^S$ the prime-to-S adele ring of K and $\mathbb{A}_S := \prod_{v\in S} K_v$. One has $\mathbb{A} = \mathbb{A}_S \times \mathbb{A}^S$.

For any reductive algebraic group G over K, an *S-arithmetic subgroup* is a subgroup of the group $G(K)$ of K-rational points which is commensurable to the intersection of $G(K)$ with an open compact subgroup U of $G(\mathbb{A}^S)$. If an S-arithmetic subgroup of G is finite, then every S-arithmetic subgroup of G is finite.

For any open compact subgroup $U \subset G(\mathbb{A}^S)$, we shall write $\mathrm{DS}(G,U)$ for the double coset space $G(K)\backslash G(\mathbb{A}^S)/U$. By the finiteness of class numbers due to Harish-Chandra and Borel (see [2]), the double coset space $\mathrm{DS}(G,U)$ is always finite.

2.2. **Mass of (G,U).** Let K and S be as above. For any group G over K with finite S-arithmetic subgroups, and any open compact subgroup $U \subset G(\mathbb{A}^S)$, we define the mass of (G,U) by

$$\mathrm{Mass}(G,U) := \sum_{i=1}^{h} |\Gamma_{c_i}|^{-1}, \tag{2.1}$$

where $c_1,\dots,c_h$ are representatives for the double coset space $\mathrm{DS}(G,U)$ and $\Gamma_{c_i} := G(K)\cap c_i U c_i^{-1}$ for $i=1,\dots,h$. Note that $\Gamma_{c_i} = \{g \in G(K) \mid g(c_iU) = c_iU\}$ and it is finite.

If $G_S := G(\mathbb{A}_S)$ is compact, then any S-arithmetic subgroup is discretely embedded into the group G_S and hence is finite. Therefore, the mass $\mathrm{Mass}(G,U)$ for (G,U) is defined for any open compact subgroup $U \subset G(\mathbb{A}^S)$.

There are examples of groups G with finite S-arithmetic subgroups whose S-component G_S needs not to be compact. For example, let D be a definite quaternion algebra over $\mathbb{Q}$ (with $S=\infty$) and $G := D^\times$ be the multiplicative group of D, viewed as an algebraic group defined over $\mathbb{Q}$. Then the group $G(\mathbb{R}) = \mathbb{H}^\times$ of $\mathbb{R}$-points, which is the group of units in the Hamilton quaternion algebra, is not compact. However, any arithmetic subgroup of $G(\mathbb{Q})$ is finite. Another example is the multiplicative group G associated to a definite central division algebra D over K with $|S|=1$.

Note that if $G_S := G(\mathbb{A}_S)$ is compact, then the group $G(K)$ is identified with a discrete subgroup in $G(\mathbb{A}^S)$ through the diagonal embedding and the quotient space $G(K)\backslash G(\mathbb{A}^S)$ is a compact topological space. This space provides a fertile ground for studying harmonic analysis. More general, we have the following equivalent statements for groups with finite S-arithmetic subgroups:

Proposition 2.1. *The following statements are equivalent.*

1. *Any S-arithmetic subgroup of $G(K)$ is finite.*
2. *The group $G(K)$ is discretely embedded into the locally topological group $G(\mathbb{A}^S)$.*
3. *The group $G(K)$ is discretely embedded into the locally topological group $G(\mathbb{A}^S)$ and the quotient space $G(K)\backslash G(\mathbb{A}^S)$ is a compact topological space.*

PROOF. See a proof in Gross [17]. ∎

In general, it is very difficult to compute the class number $h(G,U) := |\mathrm{DS}(G,U)|$. The mass $\mathrm{Mass}(G,U)$ associated to (G,U), from its definition, is a weighted class

number. It is weighted according to the extra symmetries of each double coset. On the other hand, the mass $\mathrm{Mass}(G,U)$ associated to (G,U) can be interpreted as the volume of a fundamental domain:

Lemma 2.2. *Let G be a reductive group over K with finite S-arithmetic subgroup. Then $\mathrm{Mass}(G,U) = \mathrm{vol}(G(K)\backslash G(\mathbb{A}^S)$, where we choose the Haar measure on $G(\mathbb{A}^S)$ with volume one on U and the counting measure for the discrete subgroup $G(K)$.*

Proof. Let $c_1,\dots,c_h$ be representatives for the double coset space $\mathrm{DS}(G,U)$. One has

$$G(\mathbb{A}^S) = \coprod_{i=1}^{h} G(K)c_iU.$$

Then for each class we have

$$\mathrm{volvol}(G(K)\backslash G(K)c_iU) = \frac{\mathrm{vol}(U)}{\mathrm{vol}(G(K)\cap c_iUc_i^{-1})} = |\Gamma_{c_i}|^{-1}.$$

Therefore, we have

$$\mathrm{vol}(G(K)\backslash G(\mathbb{A}^S) = \sum_{i=1}^{h} \mathrm{vol}(G(K)\backslash G(K)c_iU) = \sum_{i=1}^{h} |\Gamma_{c_i}|^{-1} = \mathrm{Mass}(G,U).$$

This proves the lemma. ∎

This interpretation of $\mathrm{Mass}(G,U)$ allows us to compare the masses $\mathrm{Mass}(G,U)$ and $\mathrm{Mass}(G,U')$ for different open compact subgroups U and U' in $G(\mathbb{A}^S)$. Precisely, one has

$$\mathrm{Mass}(G,U') = \mathrm{Mass}(G,U)[U:U'], \tag{2.2}$$

where the index $[U:U']$ is defined by

$$[U:U'] := [U:U''][U':U'']^{-1} \tag{2.3}$$

for any open compact subgroup $U''\subset U\cap U'$.

2.3. **Mass of** (D,R)**.** Let K, S, A be as above. Let D be a definite central simple algebra over K with respect to S. This means that the completion $D_v := D\otimes_K K_v$ at v for any place $v\in S$ is a central *division* algebra over K_v. In particular D is a division algebra. In the literature, definite central simple algebras are exactly those that do not satisfy the S-Eichler condition.

Let $S_D \subset V^K$ denote the finite set of ramified places for D. When D is a quaternion algebra, the condition for definiteness of D simply means that $S\subset S_D$. However, the condition $S\subset S_D$ is not sufficient to conclude that D is definite in general. One also needs to know the invariants of D.

Let R be an A-order in D. Let $\mathrm{Cl}(R)$ denote the set of equivalence classes of locally free right R-ideals. Two right R-ideals I and I' are said to be *equivalent*, which we denote by $I_1\sim I_2$, if there is an element $g\in D^\times$ such that $I' = gI$. In other words, $I_1\sim I_2$ if and only if I and I' are isomorphic as right R-modules. It is well-known that the set $\mathrm{Cl}(R)$ is always finite, and that this set can be parametrized by an adelic class space:

$$\mathrm{Cl}(R) \simeq D^\times\backslash D^\times_{\mathbb{A}^S}/\widehat{R}^\times,$$

where $\widehat{R}=\prod_{v\notin S} R_v$ ($R_v=R\otimes_A A_v$) is the profinite completion of R and $D_{\mathbb{A}^S}=D\otimes_K \mathbb{A}^S$ is the attached prime-to-S adele ring of D. Write $h(R)$ for the cardinality of $\mathrm{Cl}(R)$, the (locally free) *class number* of R.

Let $I_1,\dots,I_h$ be representatives for the ideal classes in $\mathrm{Cl}(R)$. Let R_i be the left order of I_i. Then $[R_i^\times : A^\times]$ is finite. This follows from the Dirichlet theorem that $A^\times$ is finitely generated $\mathbb{Z}$-module of rank $|S|-1$ and the following exact sequence:

$$1\to R_{i,1}^\times/A_1^\times \to R_i^\times/A^\times \to \mathrm{Nr}(R_i^\times)/\mathrm{Nr}(A^\times)\to 1,$$

where $\mathrm{Nr}: D^\times\to K^\times$ is the reduced norm. Note that the abelian groups $\mathrm{Nr}(A^\times)=(A^\times)^{\deg(D/K)}$ and $\mathrm{Nr}(R_i^\times)$ are subgroups of finite index in $A^\times$. Therefore, the quotient group $\mathrm{Nr}(R_i^\times)/\mathrm{Nr}(A^\times)$ is a finite abelian group. As each group $R_{i,1}^\times/A_1^\times$ is finite, one concludes that $R_i^\times/A^\times$ is also finite. Define the mass $\mathrm{Mass}(D,R)$ by

$$\mathrm{Mass}(D,R):=\sum_{i=1}^h [R_i^\times : A^\times]^{-1}. \tag{2.4}$$

The definition is independent of the choice of the representatives I_i.

When $|S|=1$, the group $G(K)=D^\times$ has finite S-arithmetic subgroups and hence the mass $\mathrm{Mass}(G,U)$ is also defined, where $U:=\widehat{R}^\times$. In case case, we put

$$\mathrm{Mass}^{\mathrm{u}}(D,R):=\mathrm{Mass}(G,U)=\sum_{i=1}^h |R_i^\times|^{-1}. \tag{2.5}$$

Clearly we have $\mathrm{Mass}(D,R)=|A^\times|\cdot \mathrm{Mass}^{\mathrm{u}}(D,R)$.

3. Comparison of masses

3.1. **Notation.** Let K, S and A be as in Section 1. Let D be a definite central simple algebra over K of degree n (with respect to S). Let R be an A-order in D. Let $G=D^\times$ be the multiplicative group of D, viewed as an algebraic group over K. Let Z be the center of G and $G^{\mathrm{ad}}=G/Z$ be the adjoint group of G. We have a short exact sequence of algebraic groups over K:

$$1\longrightarrow Z\longrightarrow G\xrightarrow{\mathrm{pr}} G^{\mathrm{ad}}\longrightarrow 1, \tag{3.1}$$

where pr is the natural projection morphism. Let $\mathbb{G}_{\mathrm{m}}$ denote the multiplicative group over K, and $\mathrm{Nr}: G\to \mathbb{G}_{\mathrm{m}}$ be the morphism induced from the reduced norm map $\mathrm{Nr}: D^\times\to K^\times$. Let $G_1:=\ker \mathrm{Nr}\subset G$ be the reduced norm one subgroup. We have a short exact sequence of algebraic groups over K:

$$1\longrightarrow G_1\longrightarrow G\xrightarrow{\mathrm{Nr}} \mathbb{G}_{\mathrm{m}}\longrightarrow 1. \tag{3.2}$$

The group G_1 is an inner form of SL_n and hence is semi-simple and simply-connected.

3.2. **Compare** $\mathrm{Mass}(D,R)$ **and** $\mathrm{Mass}(G^{\mathrm{ad}},U^{\mathrm{ad}})$**.** Let G be as above. For any open compact subgroup $U\subset G(\mathbb{A}^S)$, we write U^{ad} for the image $\mathrm{pr}(U)$ of U in $G^{\mathrm{ad}}(\mathbb{A}^S)$, which is an open and compact subgroup. Now let $U=\widehat{R}^\times$. Using the Hilbert Theorem 90, we have

$$G^{\mathrm{ad}}(K_v)=G(K_v)/K_v^\times \quad \text{and} \quad G^{\mathrm{ad}}(K)=D^\times/K^\times.$$

The surjective map $\mathrm{pr}: G(K_v) \to G^{\mathrm{ad}}(K_v)$ sends hyperspecial maximal open compact subgroups to those of $G^{\mathrm{ad}}(K_v)$ for almost all v. This follows from Lang's theorem asserting that any H-torsor is trivial for any connected linear algebraic group H over a finite field. So we have a surjective map of double coset spaces:

$$\mathrm{pr}: \mathrm{DS}(G,U) \to \mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) = D^\times \backslash G(\mathbb{A}^S)/\mathbb{A}^{S,\times}\widehat{R}^\times, \tag{3.3}$$

where $U^{\mathrm{ad}} = \widehat{R}^\times/\widehat{A}^\times \subset G^{\mathrm{ad}}(\mathbb{A}^S)$. Since $G^{\mathrm{ad}}(K_v) = D_v^\times/K_v^\times$ is compact for all $v \in S$, the group G^{ad} has finite S-arithmetic subgroups and hence the mass $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ is defined. We want to compare the mass $\mathrm{Mass}(D,R)$ with the mass $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$.

Let $\mathrm{Pic}(A) = \mathbb{A}^{S,\times}/K^\times\widehat{A}^\times$ be the Picard group of A and let $h_A = |\mathrm{Pic}(A)|$ be the class number of A. This group acts on $\mathrm{DS}(G,U)$ by $[a]\cdot[c] = [ca]$ for $a \in \mathbb{A}^{S,\times}$ and $c \in G(\mathbb{A}^S)$, where $[c]$ is the class in $\mathrm{DS}(G,U)$. One has the induced bijection

$$\mathrm{pr}: \mathrm{DS}(G,U)/\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}}). \tag{3.4}$$

For $c \in G(\mathbb{A}^S)$, write $[c]^{\mathrm{ad}}$ for the class in $\mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$.

By definition, we have

$$\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) = \sum_{[c]^{\mathrm{ad}} \in \mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}})} |\Gamma_c^{\mathrm{ad}}|^{-1},$$

where $\Gamma_c^{\mathrm{ad}} = G^{\mathrm{ad}}(K) \cap cU^{\mathrm{ad}}c^{-1}$. We have

$$\Gamma_c^{\mathrm{ad}} = (D^\times \cap c\widehat{R}^\times c^{-1}\mathbb{A}^{S,\times})/K^\times. \tag{3.5}$$

This group contains $(D^\times \cap c\widehat{R}^\times c^{-1}K^\times)/K^\times = R_c^\times/A^\times$ as a subgroup, where $R_c = D \cap c\widehat{R}c^{-1}$, which is also the left order of the ideal class corresponding to the class $[c]$. Therefore, the contribution of the class $[c]^{\mathrm{ad}}$ in $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ is equal to

$$|\Gamma_c^{\mathrm{ad}}|^{-1} = |R_c^\times/A^\times|^{-1}|(D^\times \cap c\widehat{R}^\times c^{-1}\mathbb{A}^{S,\times})/(D^\times \cap K^\times c\widehat{R}^\times c^{-1})|^{-1}. \tag{3.6}$$

On the group G, we have

$$\mathrm{pr}^{-1}([c]^{\mathrm{ad}}) = \{[ac]; a \in \mathbb{A}^{S,\times}\}.$$

The group $\Gamma_{ac} = D^\times \cap (ac)\widehat{R}^\times(ac)^{-1} = \Gamma_c = R_c^\times$. Therefore, the contribution of the fiber $\mathrm{pr}^{-1}([c]^{\mathrm{ad}})$ in $\mathrm{Mass}(D,R)$ is equal to

$$|R_c^\times/A^\times|^{-1}\frac{h_A}{|\mathrm{Stab}([c])|}, \tag{3.7}$$

where $\mathrm{Stab}([c])$ is the stabilizer of the class $[c]$ under the $\mathrm{Pic}(A)$-action. One has

$$[ac] = [c] \iff D^\times ac\widehat{R}^\times = D^\times c\widehat{R}^\times \iff a \in \mathbb{A}^{S,\times} \cap D^\times c\widehat{R}^\times c^{-1}.$$

and hence

$$\mathrm{Stab}([c]) \simeq (\mathbb{A}^{S,\times} \cap D^\times c\widehat{R}^\times c^{-1})/K^\times\widehat{A}^\times. \tag{3.8}$$

We now show

Lemma 3.1. *There is an isomorphism of finite abelian groups*

$$\mathrm{Stab}([c]) \simeq (D^\times \cap \mathbb{A}^{S,\times}c\widehat{R}^\times c^{-1})/(D^\times \cap K^\times c\widehat{R}^\times c^{-1}). \tag{3.9}$$

PROOF. To simply notation, put $U := c\widehat{R}^\times c^{-1}$. First of all for $a \in \mathbb{A}^{S,\times}$ we have

$$aU \cap D^\times \neq \emptyset \iff a \in \mathbb{A}^{S,\times} \cap D^\times U.$$

We now show that for $a \in \mathbb{A}^{S,\times} \cap D^\times U$, the intersection $aU \cap D^\times$ defines an element in $(\mathbb{A}^{S,\times} U \cap D^\times)/(K^\times U \cap D^\times)$. Suppose we have two elements $ax_1 = d_1$, $ax_2 = d_2$, where $x_1, x_2 \in U$ and $d_1, d_2 \in D^\times$. Then

$$(ax_1)^{-1}(ax_2) = x_1^{-1}x_2 = d_1^{-1}d_2 \in U \cap D^\times \subset K^\times U \cap D^\times.$$

Therefore, we define a map

$$\mathbb{A}^{S,\times} \cap D^\times U \to (\mathbb{A}^{S,\times} U \cap D^\times)/(K^\times U \cap D^\times), \quad a \mapsto [aU \cap D^\times].$$

We need to show that elements which go to the identity class lie in $K^\times \widehat{A}^\times$. Suppose an element $ax \in aU \cap D^\times$ lies in the identity class, i.e. $ax = ky$ for some $k \in K^\times$ and $y \in U$. Then the element $ak^{-1} = yx^{-1}$ lies in $\mathbb{A}^{S,\times} \cap U = \widehat{A}^\times$. This shows that $a \in K^\times \widehat{A}^\times$. Therefore, the above map induces a bijection

$$(\mathbb{A}^{S,\times} \cap D^\times U)/K^\times \widehat{A}^\times \simeq (\mathbb{A}^{S,\times} U \cap D^\times)/(K^\times U \cap D^\times).$$

Moreover, this is an isomorphism of finite abelian groups. Combining with the isomorphism (3.8), one obtains an isomorphism (3.9). This completes the proof of the lemma. ∎

Theorem 3.2. *Let D, R, G^{ad}, U^{ad} be as above. We have the equality*

$$\mathrm{Mass}(D, R) = h_A \cdot \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}). \tag{3.10}$$

PROOF. We have

$$\begin{aligned}
\mathrm{Mass}(D,R) &= \sum_{[c]^{\mathrm{ad}}} \sum_{[c'] \in \mathrm{pr}^{-1}([c]^{\mathrm{ad}})} |R_{c'}^\times/A^\times|^{-1} \quad \text{(by definition)} \\
&= \sum_{[c]^{\mathrm{ad}}} |R_c^\times/A^\times|^{-1} \frac{h_A}{|\,\mathrm{Stab}([c])|} \quad \text{(by (3.7))} \\
&= h_A \sum_{[c]^{\mathrm{ad}}} |R_c^\times/A^\times|^{-1} \left| \frac{(D^\times \cap c\widehat{R}^\times c^{-1} \mathbb{A}^{S,\times})}{(D^\times \cap K^\times c\widehat{R}^\times c^{-1})} \right|^{-1} \quad \text{(by Lemma 3.1)} \\
&= h_A \sum_{[c]^{\mathrm{ad}}} |\Gamma_c^{\mathrm{ad}}|^{-1} \quad \text{(by (3.6))} \\
&= h_A \cdot \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}),
\end{aligned} \tag{3.11}$$

where $[c]^{\mathrm{ad}}$ runs over all double cosets in $\mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$. This completes the proof of the theorem. ∎

Corollary 3.3. *Let R and R' be two A-order in D. Then we have*

$$\mathrm{Mass}(D, R) = \mathrm{Mass}(D, R')[\widehat{R}'^\times : \widehat{R}^\times], \tag{3.12}$$

where the index $[\widehat{R}'^\times : \widehat{R}^\times]$ is defined in (2.3).

PROOF. Since the both groups $\widehat{R}'^{\times}$ and $\widehat{R}^{\times}$ contain the center $\widehat{A}^{\times}$, one has

$$[\widehat{R}'^{\times} : \widehat{R}^{\times}] = [U'^{\mathrm{ad}} : U^{\mathrm{ad}}],$$

where $U'^{\mathrm{ad}} = \mathrm{pr}(\widehat{R}'^{\times})$ and $U^{\mathrm{ad}} = \mathrm{pr}(\widehat{R}^{\times})$. As

$$\mathrm{Mass}(G^{\mathrm{ad}}, U'^{\mathrm{ad}}) = \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})[U'^{\mathrm{ad}} : U^{\mathrm{ad}}],$$

the assertion follows immediately from Theorem 3.2. ∎

Remark 3.4. (1) When the class number h_A of A is one, the map pr in (3.3) is bijective. In this case we have the equality $|\mathrm{DS}(G,U)| = |\mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}})|$*, and the equality $\mathrm{Mass}(D,R) = \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ of different masses in Theorem 3.2 is the term-by-term equality.

(2) The action of $\mathrm{Pic}(A)$ on the double coset space $\mathrm{DS}(G,U)$ needs not to be free in general. Therefore, the class number $|\mathrm{DS}(G,U)|$ may not be equal to $h_A \cdot |\mathrm{DS}(G^{\mathrm{ad}}, U^{\mathrm{ad}})|$. To see this, let us look at the isotropic subgroup of the identity class [1] ($c = 1$ in (3.8)):

$$\mathrm{Stab}([1]) \simeq (\mathbb{A}^{S,\times} \cap D^{\times}\widehat{R}^{\times})/K^{\times}\widehat{A}^{\times}.$$

In the extreme case one considers the possibility of the equality

$$\mathbb{A}^{S,\times} \cap D^{\times}\widehat{R}^{\times} = \mathbb{A}^{S,\times}.$$

This is possible if one can find a maximal subfield L of D over K which satisfies the *Principal Ideal Theorem* (cf. Artin and Tate [1, Chapter XIII, Section 4, p.137–141]), that is, $\mathbb{A}^{S,\times} \subset L^{\times}\widehat{B}^{\times}$, where B is the integral closure of A in L. Below is an explicit example provided by F.-T. Wei.

3.3. **An example.** Let $K = \mathbb{Q}(\sqrt{10})$ and $L = K(\sqrt{-5}) = \mathbb{Q}(\sqrt{-5}, \sqrt{-2})$. Let D be the quaternion algebra over K which is ramified exactly at the two real places of K. Since L/K is inert at the real places, we can embed L into D over K. Notice that 2 and 5 are ramified in K. Let $\mathfrak{p}$ be the prime of $O_K = \mathbb{Z}[\sqrt{10}]$ lying over 5.

Claim: $\mathfrak{p} = \sqrt{10}\,O_K + 5O_K$ and $\mathfrak{p}$ is of order 2 in $\mathrm{Pic}(O_K)$.

Proof of the claim: Let $\mathfrak{q}$ be the unique prime of O_K lying over 2. Then $\sqrt{10}\,O_K = \mathfrak{p}\mathfrak{q}$ and $5O_K = \mathfrak{p}^2$. Therefore, $\mathfrak{p} = \sqrt{10}\,O_K + 5O_K$, and $\mathfrak{p}^2 = 5O_K$ is principal. We now show that $\mathfrak{p}$ is not principal. Suppose that $\mathfrak{p}$ is principal. Then there exist $x, y \in \mathbb{Z}$ such that $\mathrm{Nr}(x + y\sqrt{10}) = x^2 - 10y^2 = \pm 5$. Then $x = 5x'$ for some $x' \in \mathbb{Z}$, and $5x'^2 - 2y^2 = \pm 1 \equiv \pm 1 \pmod 5$. This implies that $-2y^2 \equiv \pm 1 \pmod 5$, which is a contradiction.

Moreover, we have

$$\mathfrak{p}O_L = \sqrt{10}\,O_L + 5O_L = \sqrt{-5}\,(\sqrt{-2}\,O_L + \sqrt{-5}\,O_L) = \sqrt{-5}\,O_L,$$

which is principal. Let R be a maximal order in D which contains O_L. Then $\mathfrak{p}R = \sqrt{-5}R$. This shows that the isotropic subgroup of the identity class [1] is

---

*This is asserted as [20, Equation (1), p. 190] and is used to prove the assertion $\mathrm{Mass}(D,R) = \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ in the proof of Theorem 1 in that paper. Both assertions are true in an accident case where $h_A = 1$ but false in general. Fortunately, Theorem 1 of [20] is correct as stated for arbitrary totally real number fields. One needs to use Corollary 3.3 first to reduce to the case where R is a maximal order. Then one applies Eichler's mass formula for totally definite quaternion algebras over totally real number fields. The index in the right hand side of (3.12) can be computed locally and one uses the local computations in [19] to furnish the proof. This fills the gaps in the proof of [20, Theorem 1]

non-trivial. As the class number $h(O_K)$ of O_K is equal to 2, this also shows that the canonical map $\mathrm{Pic}(O_K) \to \mathrm{Pic}(O_L)$, sending $[I]$ to $[IO_L]$, is the zero map.

3.4. **Comparison of** $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ **and** $\mathrm{Mass}(G_1, U_1)$**.** Recall that G_1 is the derived group of G and $U_1 := U \cap G_1(\mathbb{A}^S)$, where $U = \widehat{R}^\times$. Let $\widetilde{R}$ be a maximal A-order in D containing R. Put $\widetilde{U} := (\widetilde{R} \otimes_A \widehat{A})^\times$ and $\widetilde{U}_1 := \widetilde{U} \cap G_1(\mathbb{A}^S)$. We compare the masses $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ and $\mathrm{Mass}(G_1, U_1)$. Using the interpretation of masses as the volume of fundamental domains (Lemma 2.2), one first has

$$\begin{aligned} \mathrm{Mass}(G_1, U_1) &= \mathrm{Mass}(G_1, \widetilde{U}_1)[\widetilde{U}_1 : U_1], \\ \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) &= \mathrm{Mass}(G^{\mathrm{ad}}, \widetilde{U}^{\mathrm{ad}})[\widetilde{U}^{\mathrm{ad}} : U^{\mathrm{ad}}]. \end{aligned} \tag{3.13}$$

From this we see that the comparison of these two masses depends on U and can be reduced to the case where R is a maximal A-order. Put

$$c(S, U) := \frac{\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})}{\mathrm{Mass}(G_1, U_1)}. \tag{3.14}$$

Lemma 3.5. *One has*

$$c(S, U) = c(S, \widetilde{U}) \cdot [\widehat{A}^\times : \mathrm{Nr}(U)], \tag{3.15}$$

where $\mathrm{Nr} : G(\mathbb{A}^S) \to \mathbb{A}^{S,\times}$ *is the reduced norm map.*

Proof. Using the relation (3.13) we get

$$c(S, U) = c(S, \widetilde{U}) \cdot \frac{[\widetilde{U}^{\mathrm{ad}} : U^{\mathrm{ad}}]}{[\widetilde{U}_1 : U_1]}. \tag{3.16}$$

Since both U and $\widetilde{U}$ contain the center $\widehat{A}^\times$, one has $[\widetilde{U}^{\mathrm{ad}} : U^{\mathrm{ad}}] = [\widetilde{U} : U]$. Using the following short exact sequences

$$1 \longrightarrow U_1 \longrightarrow U \longrightarrow \mathrm{Nr}(U) \longrightarrow 1,$$

$$1 \longrightarrow \widetilde{U}_1 \longrightarrow \widetilde{U} \longrightarrow \mathrm{Nr}(\widetilde{U}) = \widehat{A}^\times \longrightarrow 1,$$

one easily shows that $[\widetilde{U} : U] = [\widetilde{U}_1 : U_1] \cdot [\widehat{A}^\times : \mathrm{Nr}(U)]$. The equality (3.15) then follows from the relation (3.16). This proves the lemma. ∎

Theorem 3.6. *Let the notations be as above. Assume that $S = \infty$ and that R is a maximal A-order.*

(1) *If K is a totally real number field, then*

$$\mathrm{Mass}(D, R) = h_A \cdot \frac{\tau(G^{\mathrm{ad}})}{2^{[K:\mathbb{Q}]}} \cdot (-1)^{[K:\mathbb{Q}]} \zeta_K(-1) \cdot \prod_v (q_v - 1), \tag{3.17}$$

$$\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) = \frac{\tau(G^{\mathrm{ad}})}{2^{[K:\mathbb{Q}]}} \cdot (-1)^{[K:\mathbb{Q}]} \zeta_K(-1) \cdot \prod_v (q_v - 1), \tag{3.18}$$

and

$$\mathrm{Mass}(G_1, U_1) = \frac{\tau(G_1)}{2^{[K:\mathbb{Q}]}} \cdot (-1)^{[K:\mathbb{Q}]} \zeta_K(-1) \cdot \prod_v (q_v - 1), \tag{3.19}$$

where v runs through all ramified non-Archimedean places of K for D, $\tau(G^{\mathrm{ad}})$ and $\tau(G_1)$ are the Tamagawa numbers of G^{ad} and G_1, respectively.

(2) *If K is a global function field, then*

$$\mathrm{Mass}(D,R) = h_A \cdot \frac{\tau(G^{\mathrm{ad}})}{n} \cdot \prod_{i=1}^{n-1} \zeta_K(-i) \cdot \prod_{v \in S_D} \lambda_v, \tag{3.20}$$

$$\mathrm{Mass}(G^{\mathrm{ad}},U^{\mathrm{ad}}) = \frac{\tau(G^{\mathrm{ad}})}{n} \cdot \prod_{i=1}^{n-1} \zeta_K(-i) \cdot \prod_{v \in S_D} \lambda_v, \tag{3.21}$$

and

$$\mathrm{Mass}(G_1,U_1) = \tau(G_1) \cdot \prod_{i=1}^{n-1} \zeta_K(-i) \cdot \prod_{v \in S_D} \lambda_v, \tag{3.22}$$

where S_D is the set of all (non-Archimedean) places where D is ramified,

$$\lambda_v = \prod_{1 \le i \le n-1,\, d_v \nmid i} (q_v^i - 1) \tag{3.23}$$

and d_v is the index of $D_v := D \otimes_K K_v$.

PROOF. (1) The formulas for $\mathrm{Mass}(D,R)$ and $\mathrm{Mass}(G_1,U_1)$ are due to Eichler [10]; also see [26, Chapter V]. The formula for $\mathrm{Mass}(G^{\mathrm{ad}},U^{\mathrm{ad}})$ follows from Eichler's formula for $\mathrm{Mass}(D,R)$ and Theorem 3.2.

(2) The formula for $\mathrm{Mass}(D,R)$ is obtained by Denert and Van Geel [7] and also by Wei and the author [30, Theorem 1.1]. The formula for $\mathrm{Mass}(G_1,U_1)$ follows from the relation $\mathrm{Mass}(D,R) = h_A \cdot \mathrm{Mass}(G_1,U_1)$; see [35, Eq. (3), p. 907]. The formula for $\mathrm{Mass}(G^{\mathrm{ad}},U^{\mathrm{ad}})$ follows from the formula for $\mathrm{Mass}(D,R)$ and Theorem 3.2. This completes the proof of the theorem. ∎

Remark 3.7. In the function field case with $|S|=1$ the notation $\mathrm{Mass}(D,R)$ in [30] is defined to be the un-normalized mass $\mathrm{Mass}^{\mathrm{u}}(D,R)$ (2.5) in this paper, which is $(q-1)^{-1}$ times the mass $\mathrm{Mass}(D,R)$ in this paper.

Theorem 3.8. *Let the notations be as above. Assume that R is a maximal A-order. We have*

$$\begin{aligned} \mathrm{Mass}(G^{\mathrm{ad}},U^{\mathrm{ad}}) &= c^{\mathrm{ad}} \cdot L \cdot \prod_v \lambda_v, \\ \mathrm{Mass}(G_1,U_1) &= c_1 \cdot L \cdot \prod_v \lambda_v, \end{aligned} \tag{3.24}$$

where $L := |\prod_{i=1}^{n-1} \zeta_K(-i)|$, λ_v is as (3.23), $c^{\mathrm{ad}} := \tau(G^{\mathrm{ad}})/n^{|S|}$ and

$$c_1 := \begin{cases} \tau(G_1) & \text{if } K \text{ is a function field;} \\ \tau(G_1) \cdot 2^{-[K:\mathbb{Q}]} & \text{if } K \text{ is a number field.} \end{cases} \tag{3.25}$$

Proof. We have

$$
\begin{aligned}
\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) &= \frac{\mathrm{vol}(G^{\mathrm{ad}}(K)\backslash G^{\mathrm{ad}}(\mathbb{A}^S))}{\mathrm{vol}(U)} \\
&= \frac{\mathrm{vol}(G^{\mathrm{ad}}(K)\backslash G^{\mathrm{ad}}(\mathbb{A}^\infty))}{\mathrm{vol}(\prod_{v\in S-\infty} G^{\mathrm{ad}}(O_v)\cdot U)} \cdot \prod_{v\in S-\infty} \left[\frac{\mathrm{vol}(G^{\mathrm{ad}}(K_v))}{\mathrm{vol}(G^{\mathrm{ad}}(O_v))}\right]^{-1} \\
&= \frac{1}{n^{|S|-|\infty|}} \frac{\mathrm{vol}(G^{\mathrm{ad}}(K)\backslash G^{\mathrm{ad}}(\mathbb{A}^\infty))}{\mathrm{vol}(\prod_{v\in S-\infty} G^{\mathrm{ad}}(O_v)\cdot U)} \\
&= \frac{1}{n^{|S|-|\infty|}} \cdot \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}, S=\infty).
\end{aligned} \tag{3.26}
$$

Here $G^{\mathrm{ad}}(O_v) = O_{D_v}^\times / O_{K_v}^\times$ where O_{D_v} is the valuation ring in the division algebra D_v, and we use the isomorphism $G^{\mathrm{ad}}(K_v)/G^{\mathrm{ad}}(O_v) \simeq \mathbb{Z}/n\mathbb{Z}$. The computation above reduces to the case where $S=\infty$. Using the formulas (3.18) and (3.21) we compute

$$
c^{\mathrm{ad}} = \frac{1}{n^{|S|-|\infty|}} \cdot \frac{\tau(G^{\mathrm{ad}})}{n^{|\infty|}} = \frac{\tau(G^{\mathrm{ad}})}{n^{|S|}}.
$$

This settles the formula for $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$. Using $G_1(K_v) = G_1(O_v)$ for $v \in S$, the same computation as (3.26) shows that $\mathrm{Mass}(G_1, U_1)$ is independent of S. Therefore, the formula for $\mathrm{Mass}(G_1, U_1)$ is given by (3.19) and (3.22), respectively. This completes the proof of the theorem. ∎

We now show the following comparison result.

Corollary 3.9. *Let the notations be as above and let R be any A-order in D. Then we have*

$$
\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) = c(S,U) \cdot \mathrm{Mass}(G_1, U_1), \tag{3.27}
$$

where

$$
c(S,U) := \begin{cases} n^{-(|S|-1)}[\widehat{A}^\times : \mathrm{Nr}(U)] & \text{if } K \text{ is a function field;} \\ 2^{-(|S|-|\infty|-1)}[\widehat{A}^\times : \mathrm{Nr}(U)] & \text{if } K \text{ is a number field.} \end{cases} \tag{3.28}
$$

Proof. When R is a maximal order, we compute by Theorem 3.8

$$
c(S,\widetilde{U}) = \begin{cases} n^{-(|S|-1)} & \text{if } K \text{ is a function field;} \\ 2^{-(|S|-|\infty|-1)} & \text{if } K \text{ is a number field.} \end{cases} \tag{3.29}
$$

The statement then follows from Lemma 3.3. ∎

4. Mass formulas for arbitrary orders R

In the previous sections we deduce the formulas for $\mathrm{Mass}(D,R)$, $\mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}})$ and $\mathrm{Mass}(G_1, U_1)$ in the case where the order R considered is maximal. In this section we consider the general case. In the view of the comparison results Theorem 3.2 and Corollary 3.9, it suffices to derive the formula for $\mathrm{Mass}(D,R)$ with an arbitrary A-order R.

4.1. **More notations.** Let A be any Dedekind domain and let K be the fraction field of R. Let V be a finite-dimensional K-vector space. For any two (full) A-lattices X_1 and X_2, let $\chi(X_1, X_2)$ be the unique fractional ideal of A that is characterized by the following properties (See Serre [24, Chapter III, Section 1]):

- If $X_1/X_2 \simeq A/\mathfrak{p}$ for a non-zero prime ideal $\mathfrak{p} \subset A$, then $\chi(X_1, X_2) = \mathfrak{p}$.
- $\chi(X_1, X_2) = \chi(X_2, X_1)^{-1}$ for any two A-lattices X_1 and X_2 in V.
- $\chi(X_1, X_2)\chi(X_2, X_3) = \chi(X_1, X_3)$ for any three A-lattices X_1, X_2 and X_3 in V.

When K is a global field, we define $|I| := |A/I|$ for any non-zero integral ideal $I \subset A$ and extend the definition to fractional ideals by

$$|I_1 I_2^{-1}| = |I_1||I_2|^{-1}$$

for non-zero integral ideals I_1 and I_2 of A. In this case let $\widehat{A}$ denote the finite completion of A and $\widehat{K} := \widehat{A} \otimes_A K$. Put $\widehat{X} := X \otimes_A \widehat{A}$ and $\widehat{V} := V \otimes_K \widehat{K}$. Then for any Haar measure on $\widehat{V}$ one has

$$|\chi(X_1, X_2)| = \frac{\mathrm{vol}(\widehat{X}_1)}{\mathrm{vol}(\widehat{X}_2)}. \tag{4.1}$$

Now we define the *discriminant* of an A-lattice with respect to a bilinear form on V (for any Dedekind domain A). Let $T : V \times V \to K$ be a non-degenerate K-bilinear map. Put $n = \dim_K V$. For any K-basis $E = \{e_1, e_2, \dots, e_n\}$ of V, the *discriminant of* E with respect to T is defined to be

$$D_T(E) := \det(T(e_i, e_j)) \in K. \tag{4.2}$$

Let X be an A-lattice in V. The *discriminant of* X with respect to T is defined to be the fractional ideal

$$\mathfrak{d}_T(X) := (D_T(E); E) \subset K \tag{4.3}$$

where E runs through all K-bases E such that $A\langle E\rangle \subset X$, where $A\langle E\rangle$ is the A-submodule generated by E. Computation of the discriminants can be reduced to the local computation, namely, we have

$$\mathfrak{d}_T(X) \otimes_A A_{\mathfrak{p}} = \mathfrak{d}_T(X_{\mathfrak{p}}), \quad X_{\mathfrak{p}} := X \otimes_A A_{\mathfrak{p}}, \tag{4.4}$$

where $A_{\mathfrak{p}}$ is the completion of A at the non-zero prime ideal $\mathfrak{p}$.

Let X_1 and X_2 be two A-lattices in V. Then we have the formula [24, Chap. III, § 2, Proposition 5, p. 49]

$$\mathfrak{d}_T(X_2) = \mathfrak{d}_T(X_1)\chi(X_1, X_2)^2. \tag{4.5}$$

In particular, if $X_2 \subset X_1$ then $\mathfrak{d}_T(X_2) = \mathfrak{d}_T(X_1)\mathfrak{a}^2$, where $\mathfrak{a} = \chi(X_1, X_2)$, an integral ideal of A.

Now we define the *reduced discriminant* of an A-lattice in a *central simple algebra* over K; some authors simply call this the *discriminant* of the lattice. Let D be a central simple K-algebra and X be an A-lattice in D. Let $T : D \times D \to K$ be the non-degenerate K-bilinear form defined by

$$T(x, y) := \mathrm{Tr}(x \cdot y),$$

where $\mathrm{Tr} : D \to K$ is the reduced trace from D to K. Then $\mathfrak{d}_T(X)$ is defined and it can be shown to be the square of a unique fractional ideal $\mathfrak{a}$ in K. The *reduced discriminant of* X, denoted by $\mathfrak{d}(X)$, is defined to this fractional ideal $\mathfrak{a}$, namely,

the square root of $\mathfrak{d}_T(X)$. It is easy to see that the map $X \mapsto \mathfrak{d}(X)$ commutes with finite etale base changes and localizations. Namely, if B is a finite etale extension or a localization of A then one has

$$\mathfrak{d}(X \otimes_A B) = \mathfrak{d}(X) \otimes_A B. \tag{4.6}$$

4.2. **Computation of** $\mathrm{Mass}(D, R)$**.** We continue with our notations K,A, S, D, R as in the previous sections and aim at computing the mass $\mathrm{Mass}(D, R)$. Now R stands for an arbitrary A-order in D. Let $\widetilde{R}$ be a maximal A-order in D containing R. The masses $\mathrm{Mass}(D, \widetilde{R})$ and $\mathrm{Mass}(D, R)$ differ by the factor

$$\prod_{v \notin S} [\widetilde{R}_v^\times : R_v^\times], \tag{4.7}$$

Put $\kappa(R_v) := R_v/\mathrm{rad}(R_v)$, where $\mathrm{rad}(R_v)$ denotes the Jacobson radical of R_v.

Lemma 4.1.

(1) *We have*

$$[\widetilde{R}_v : R_v] = \frac{|\mathfrak{d}(R_v)|}{|\mathfrak{d}(\widetilde{R}_v)|}. \tag{4.8}$$

(2) *We have*

$$[\widetilde{R}_v^\times : R_v^\times] = \frac{|\mathfrak{d}(R_v)|}{|\mathfrak{d}(\widetilde{R}_v)|} \cdot \frac{|\kappa(\widetilde{R}_v)^\times|/|\kappa(\widetilde{R}_v)|}{|\kappa(R_v)^\times|/|\kappa(R_v)|}. \tag{4.9}$$

Proof. (1) We have $[\widetilde{R}_v : R_v] = |\chi(\widetilde{R}_v, R_v)|$ from (4.1) and $\mathfrak{d}(R_v) = \mathfrak{d}(\widetilde{R}_v)\chi(\widetilde{R}_v, R_v)$. Then we get $|\mathfrak{d}(R)| = |\mathfrak{d}(\widetilde{R}_v)| \cdot [\widetilde{R}_v : R_v]$ and (4.8).

(2) For any Haar measure on D_v we have

$$[\widetilde{R}_v^\times : R_v^\times] = \frac{\mathrm{vol}(\widetilde{R}_v^\times)}{\mathrm{vol}(R_v^\times)} = \frac{\mathrm{vol}(\widetilde{R}_v)}{\mathrm{vol}(R_v)} \cdot \frac{|\kappa(\widetilde{R}_v)^\times|/|\kappa(\widetilde{R}_v)|}{|\kappa(R_v)^\times|/|\kappa(R_v)|}$$

Then we obtain the formula (4.9) from the formula (4.8). This completes the proof of the lemma. ∎

For any non-Archimedean place $v \in S$, we put $R_v := O_{D_v}$, where O_{D_v} is the unique maximal order of the division algebra D_v. For any non-Archimedean place v and any A_v-order R_v in D_v, we define

$$\lambda_v(R_v) := \frac{|\mathfrak{d}(R_v)|}{|\kappa(R_v)^\times|/|\kappa(R_v)|} \cdot \prod_{1 \le i \le n} (1 - q_v^{-i}). \tag{4.10}$$

Now we prove the following formula.

Theorem 4.2. *Let the notations K, S, A, D and R be as above. We have*

$$\mathrm{Mass}(D, R) = h_A \cdot c^{\mathrm{ad}} \cdot L \cdot \prod_v \lambda_v(R_v), \tag{4.11}$$

where c^{ad} and L are the same as in Theorem 3.8, $\lambda_v(R_v)$ is defined in (4.10) and v runs through all non-Archimedean places of K.

PROOF. By Theorems 3.2 and 3.8 we have

$$\mathrm{Mass}(D,R) = h_A \cdot c^{\mathrm{ad}} \cdot L \cdot \prod_v (\lambda_v \cdot [\widetilde{R}_v^\times : R_v^\times]), \tag{4.12}$$

where λ_v is defined in (3.23). Thus, it suffices to check

$$\lambda_v \cdot [\widetilde{R}_v^\times : R_v^\times] = \lambda_v(R_v). \tag{4.13}$$

The left hand side of (4.13) is equal to (using Lemma 4.1)

$$\lambda_v \cdot \frac{|\mathfrak{d}(R_v)|}{|\mathfrak{d}(\widetilde{R}_v)|} \cdot \frac{|\kappa(\widetilde{R}_v)^\times|/|\kappa(\widetilde{R}_v)|}{|\kappa(R_v)^\times|/|\kappa(R_v)|}. \tag{4.14}$$

Suppose $D_v = \mathrm{Mat}_m(\Delta)$, where Δ is a central division algebra with index d, thus $n = dm$. We compute the part

$$\begin{aligned} &\lambda_v \cdot \frac{1}{|\mathfrak{d}(\widetilde{R}_v)|} \cdot |\kappa(\widetilde{R}_v)^\times|/|\kappa(\widetilde{R}_v)| \\ &= \prod_{1\le i\le n-1, d\nmid i} (q_v^i - 1) \cdot \frac{1}{q_v^{m^2\cdot d(d-1)/2}} \cdot \prod_{1\le j\le m} (1 - q_v^{-dj}) \\ &= \prod_{1\le i \le n} (1 - q_v^{-i}). \end{aligned} \tag{4.15}$$

This verifies the equality (4.13) and completes the proof of the theorem. ∎

In the rest of this section we restrict to the case $n = 2$. If the order R_v is not isomorphic to $\mathrm{Mat}_2(A_v)$, then define the *Eichler symbol* $e(R_v)$ by

$$e(R_v) = \begin{cases} 1 & \text{if } \kappa(R_v) = \kappa(v) \times \kappa(v); \\ -1 & \text{if } \kappa(R_v) \text{ is a quadratic field extension of } \kappa(v); \\ 0 & \text{if } \kappa(R_v) = \kappa(v). \end{cases} \tag{4.16}$$

Corollary 4.3. *Let K, S, A, D, R be as in Theorem 4.2 and assume that $n = 2$. Then we have*

$$\mathrm{Mass}(D,R) = \frac{h_A |\zeta_K(-1)|}{2^{|S|-1}} \prod_{v\in S_R} |\mathfrak{d}(R_v)| \frac{(1-q_v^{-2})}{(1-e(R_v)q_v^{-1})}. \tag{4.17}$$

where S_R consists of all non-Archimedean places v of K such that either v is ramified in D or R_v is not maximal.

PROOF. By Theorem 4.2, it suffices to check

$$\frac{|\kappa(R_v)^\times|}{|\kappa(R_v)|} = (1-q_v^{-1})(1-e(R_v)q_v^{-1}). \tag{4.18}$$

But this is clear. ∎

When K is a totally real number field and $S = \infty$, the set of Archimedean places of K, Corollary 4.3 is obtained first by Körner (see [20, Theorem 1], also see [19] for the computation). Körner's formula is used by Brzezinski [3] to solve the class number one problem for orders in all definite quaternion algebras over $\mathbb{Q}$.

5. Mass formulas for types of orders

We keep the notations K, S, A, D, R as before. Let $\mathcal{R}$ be the *genus* of R, that is, it consists of all A-orders in D which are isomorphic to R locally everywhere. A *type* of R is a $D^\times$-conjugacy class of orders in $\mathcal{R}$. The set of $D^\times$-conjugacy classes of orders in $\mathcal{R}$ is denoted by $T(R)$. This is a finite set and its cardinality $|T(R)|$, denoted by $t(R)$, is called the *type number* of R.

Definition 5.1. Notations as above. Let $\{R_1, \dots, R_t\}$ be a set of A-orders representing the $D^\times$-conjugacy classes in $\mathcal{R}$. Define the *mass the types of R* by

$$\mathrm{Mass}(T(R)) := \sum_{i=1}^{t} [N(R_i) : K^\times]^{-1}, \tag{5.1}$$

where $N(R_i)$ is the normalizer of R_i in $D^\times$.

We know that there is a natural bijection

$$T(R) \simeq D^\times \backslash G(\mathbb{A}^S)/\mathcal{N}(\widehat{R}), \tag{5.2}$$

where $\mathcal{N}(\widehat{R})$ is the normalizer of $\widehat{R}$ in $\widehat{D}^\times = G(\mathbb{A}^S)$.

The following result evaluates $\mathrm{Mass}(T(R))$. In the computation, one also shows that each term $[N(R_i) : K^\times]$ is finite so that $\mathrm{Mass}(T(R))$ is defined.

Theorem 5.2. *Let the notations be as above. We have*

$$\mathrm{Mass}(T(R)) = c^{\mathrm{ad}} \cdot L \cdot \prod_v \lambda_v(R_v) \cdot [\mathcal{N}(\widehat{R}) : \mathbb{A}^{S,\times} \widehat{R}^\times], \tag{5.3}$$

where c^{ad}, L, and $\lambda_v(R_v)$ are as in Theorem 4.2.

Proof. Let $\mathcal{N}^{\mathrm{ad}}$ denote the image of the open subgroup $\mathcal{N}(\widehat{R}) \subset G(\mathbb{A}^S)$ in $G^{\mathrm{ad}}(\mathbb{A}^S)$. We now show

$$\mathrm{Mass}(T(R)) = \mathrm{Mass}(G^{\mathrm{ad}}, \mathcal{N}^{\mathrm{ad}}). \tag{5.4}$$

Let $c_1, \dots, c_t \in G(\mathbb{A}^S)$ be representatives for the double coset space in (5.2). For each $i = 1, \dots, t$, put

$$\Gamma_i^{\mathrm{ad}} := G^{\mathrm{ad}}(K) \cap c_i \mathcal{N}^{\mathrm{ad}} c_i^{-1}, \quad \text{and} \quad \Gamma_i := G(K) \cap c_i \mathcal{N}(\widehat{R}) c_i^{-1}. \tag{5.5}$$

It is clear that $\Gamma_i^{\mathrm{ad}} = \Gamma_i/K^\times$. So it suffices to show that $\Gamma_i = N(R_i)$. Notice $R_i = D \cap c_i \widehat{R} c_i^{-1}$, so $\widehat{R}_i = c_i \widehat{R} c_i^{-1}$. Let $x \in \Gamma_i$. Then $x = c_i y c_i^{-1}$ for some $y \in \mathcal{N}(\widehat{R})$. Therefore, $c_i^{-1} x c_i \in \mathcal{N}(\widehat{R})$. This gives $x(c_i \widehat{R} c_i^{-1})x^{-1} = (c_i \widehat{R} c_i^{-1})$. Therefore,

$$x \in \Gamma_i \iff x(\widehat{R}_i)x^{-1} = \widehat{R}_i,$$

and hence $\Gamma_i = N(R_i)$. This shows (5.4).

Using (5.4), we have $\mathrm{Mass}(T(R)) = \mathrm{Mass}(G^{\mathrm{ad}}, U^{\mathrm{ad}}) \cdot [\mathcal{N}^{\mathrm{ad}} : U^{\mathrm{ad}}]$. Then the formula (5.3) follows from Theorems 4.2 and 3.2 and $[\mathcal{N}^{\mathrm{ad}} : U^{\mathrm{ad}}] = [\mathcal{N}(\widehat{R}) : \mathbb{A}^{S,\times} \widehat{R}^\times]$. This completes the proof of the theorem. ∎

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